Dynamics of a Spinning Top

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December 18, 1998

Introduction

We have all seen and experienced some form of gyroscopic motion but probably have never given the phenomena much thought. What is a gyroscope? A broad definition would be, "a solid body capable of rotating at high angular velocity about an instantaneous axis which always passes through a fixed point" which sounds like that toy top you may have enjoyed as a child. The interesting thing about the familiar toy top is that it incorporates all those rotational motion elements that we might have studied in a typical engineering dynamics class. Now imagine the top is spinning on a horizontal surface. We notice that it moves around, but how does it do this? The explanation for this motion might look very straight forward as simply being due to the steady downward force of gravity acting on the top, Well this is only part of the story! Actually torque and angular momentum have significant roles as we will see. The top movement or skewing around of the spin axis is called precession and is dependent on the direction of the top and on which end the force is applied.

History of Tops

Spinning tops have been around so long that no one knows who spun the first one. It is likely that the first spinning top was a nut or acorn spun by a curious child. It seems likely that natural curiosity began the craft of top spinning in various places around the world. This might have happened because spinning objects are so fascinating. They are fun to play with and easy to experiment with. The Chinese have spun *tsa lin* (tops) and the *ko-en-gen* (*Diabolo*) for centuries. The world's largest top is located in China and it weighs about 280 kg. In Japan, *koma asobi* (top spinning) has been enjoyed by both adults and children for centuries. The *Dreidel* is a top used to play a traditional Jewish Hanukkah game that dates back over 2000 years. By the 1700's, Europe was introduced to both the spinning top and the *Diabolo*. Villages in Shakespear's day sometimes kept a large spinning top in the town square. On cold days villagers could spin the top as exercise and to warm themselves. The spinning top was also one of the earliest toy patents granted by the United States Patent Office. Tops were among the first toys patented in the United States. There are various kinds of spinning tops. A few have already been mentioned.

In the United States, most people are familiar with the large plunger top. The mechanism inside the top causes it to spin when the plunger is pumped up and down. The finger top (usually very inexpensive) begins spinning by twirling it between the thumb and fingers. Flip tops are designed to flip over when they are spun! The whipping top was more popular in Europe. It is spun by hitting it with a lash or whip. Mechanical springs are used in some tops to wind up the top and let it go spinning when the spring is released. The peg top is spun by winding a string or cord around the top and throwing the top to unwind the string and make the top spin.

General Equations [1, 2]

Consider the rotational motion of the top shown below. It is assumed to spin without friction such that the point O on the axis of symmetry is fixed. The only external moment about O is that due to the constant gravitational force mg acting through its center of mass at C.

Let us analyze the motion using Lagrange's equations with Eulerian angles as the co-ordinates. If we choose the fixed point O as the reference point, then the total kinetic energy can be written in terms of Euler angle rates. Noting that $I_{xx} = I_a$ and $I_{yy} = I_{zz} = I_t$ for this case of symmetry about the x-axis, we see that:

$$T = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) \tag{1}$$

$$V = mgl \sin\theta \tag{2}$$

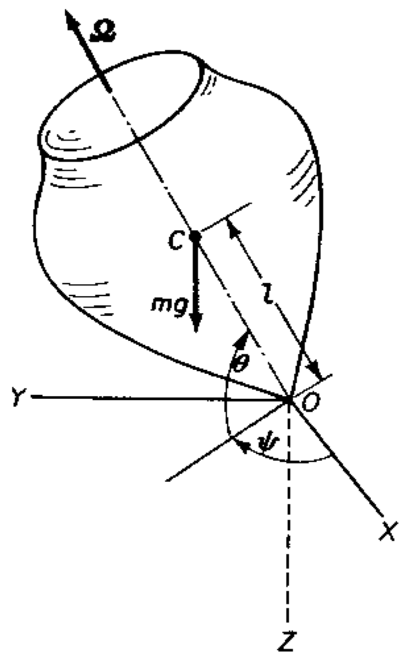

Figure 1: A symmetric top top with a fixed point at O

$$L = T - V = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) - mgl \sin\theta \tag{3}$$

The generalized momenta are:

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = -I_a \Omega \sin\theta + I_t \dot{\psi} \cos^2\theta \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = I_t \dot{\theta} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = I_a \Omega \end{aligned} \tag{4}$$

where the total spin Ω is given by

$$\Omega = \dot{\phi} - \dot{\psi} \sin\theta$$

The standard form of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

is applied together with Eq. (4) to obtain

$$\begin{aligned} \tfrac{dp_\psi}{dt} &= 0 \\ \tfrac{dp_\phi}{dt} &= 0 \end{aligned} \tag{5}$$

from which we see that both p_ψ and p_ϕ are constant. Hence we find that Ω is constant for this case where there is no applied moment about the symmetry axis. Also, the precession rate $\dot{\psi}$ can be obtained from Eq. (4) with the following result:

$$\dot{\psi} = \frac{p_\psi + I_a \Omega \sin\theta}{I_t \cos^2\theta} \tag{6}$$

Now, let us use the principle of conservation of energy to obtain an integral of the θ equation of motion. From Eqs. (1) and (2), we see that the total energy is

$$E = T + V = \frac{1}{2} I_a \Omega^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) + mgl\sin\theta \tag{7}$$

where $\Omega = \dot{\phi} - \dot{\psi}\sin\theta$ is constant. It follows that the total energy minus the kinetic energy associated with the total spin Ω is also a constant. Calling this quantity E', we can write

$$E' = E - \frac{1}{2} I_a \Omega^2 = \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) + mgl\sin\theta \tag{8}$$

Substituting for $\dot{\psi}$ from Eq. (6) and solving for $\dot{\theta}^2$, we obtain

$$\dot{\theta}^2 = \frac{2E'}{I_t} - \left(\frac{p_\psi + I_a \Omega \sin\theta}{I_t \cos^2\theta}\right) - \frac{2mgl}{I_t}\sin\theta \tag{9}$$

Note that θ is the only variable on the right-hand side of this equation. Thus we can see from Eqs. (6) and (9) that the precession rate $\dot{\psi}$ and the nutation rate $\dot{\theta}$ can be written as functions of θ alone for any given case. Solving this system of equations thus describes the complete motion of a simple spinning top.

The Tippe Top

What was described in the previous section was one of the simplest form of top motion. This section discusses the motion of a very commonly available

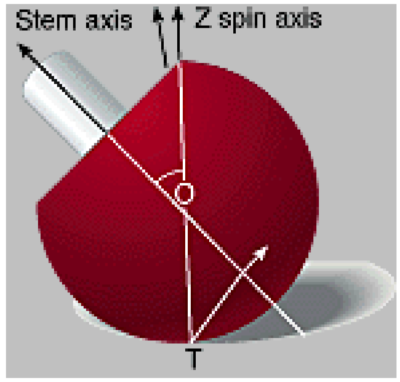

Figure 2: A Tippe Top

top, called "The Tippe Top" (shown in figure above). This is a small spinning top which has a low center of gravity. When you give it a push, it readily restores itself to the upright position. The top can be made to spin by twisting the stem between the finger and thumb. The top then precesses with the axle getting lower and lower until it flips over and continues to spin in an upturned position. When the spinning stops the top returns to its original stable state. This motion can be described rigorously only by a quite complex set of mathematical equations, but a simplified description in broad terms may suffice. The Tippe top, when spinning, refuses to sit on its rounded end, but flips over and rotates on its stem. This inversion is remarkable for two reasons: the center of mass is raised in the process of inversion, and the direction of rotation in respect to the fixed-body coordinates is reversed as the top turns over.

Analysis of this unexpected behavior has, over the years, attracted the attention of a number of very famous scientists, ranging from William Thomson in the late 19th century to Niels Bohr and Wolfgang Pauli in this century. Indeed, there is a wonderful photograph in the American Institute of Physics's Niels Bohr Library of Pauli and Bohr watching an inverting Tippe top. The first accurate modern explanations of the top's behavior date from the 1950s, when they were put forward independently by Braams, Hughenholz and Pliskin. But possibly the most rigorous analysis of the top's mechanics is that by the American physicist Richard Cohen at the Massachusetts Institute of Technology in 1977. He also developed computer-generated solutions of the equations of motion.

The behavior of the Tippe top can be described for the non-mathematician in fairly simple terms by looking at the earlier figure. When the top is set spinning the low center of mass causes the center to be centrifugally displaced from the spin axis (Z), which remains perpendicular to the surface on which the top is spinning. The angular momentum component along Z remains dominant before and after inversion, although in respect of the solid-body coordinates of the top, the direction of rotation has been reversed. During the inversion the center of mass of the top is raised and its rotational kinetic energy is reduced, providing the potential energy to raise the center of mass. Thus the total angular velocity and the total angular momentum are reduced during the inversion process. This process requires the action of a torque, but this torque cannot be provided by gravity or the normal forces exerted at the point of contact with the surface (T). The explanation lies in the presence of sliding frictional forces between the round bottom of the top and the surface on which it is spinning. These forces arise through the centrifugal displacement of the center of mass when the top is set spinning. If the initial spin velocity imparted to the top when it is set in motion is sufficiently high, nutation and oscillations of the Tippe top's motion eventually result in the edge of the stem making contact with the surface and the top then rising to the inverted position. This behavior of raising the center of mass also occurs in a normal whip-top. It is not normally appreciated that this occurs because the whip-top does not perform the spectacular inversion that is so obvious in the case of the Tippe top. It merely rises up from lying on its side into the vertical position.

Conclusion

At this point it should be easy to see just how involved the rotational motion of a simple top can really be. The results from this analysis also have heavy applications in the analysis of other systems, such as gyroscopes and spinning projectiles. Gyroscopes have a wide variety of use in today's modern technological world. They are an important features present in the navigational equipment of ships, submarines, aircraft and spacecraft. The propeller of an airplane due to its direction of rotation causes gyroscopic action. The earth and other heavenly bodies act as enormous gyroscopes/tops and are considered to be ideal in man's concept of perfect gyroscopic motion. And finally, this article highlights how beautiful the subject of dynamics can be to be able to give such precise explanations for fascinating and complex motions like the top which never fail to capture attention.

References

[1] Michèle Audin. *Spinning Tops.* Cambridge University Press, 1996.

[2] Donald T. Greenwood. *Priniciples of Dynamics.* Prentice Hall, 1988.